Lie Point Symmetries for Nonlinear Beam Equation

by

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Abstract

This thesis studies a family of nonlinear fourth-order wave equations

$$u_{tt} + \left(K'(u_{xx})\right)_{xx} = 0,$$

modelling transverse vibrations of thin elastic beams with a general constitutive function K. The classical Euler–Bernoulli beam equation is recovered when K is quadratic.

We carry out a complete Lie point symmetry classification for the scalar equation and for three equivalent formulations obtained by introducing velocity and momentum potentials V and W through $V_x = U_t$ and $W_x = tU_t - U$. Each formulation lives in a progressively larger jet space, and the enlarged fibre dimension systematically increases the generic symmetry algebra: from seven dimensions (U-system) to eight (UV) to nine (UVW). The analysis, performed with the `GeM` package in Maple, identifies every nonlinear constitutive function K that enlarges the symmetry algebra beyond the generic case. Distinguished nonlinearities include power law, exponential, reciprocal, logarithmic, and fractional-power forms. The reciprocal-cube model $K(z) = -256A/\left(3(B+z)^3\right) + C$ is shown to be the unique maximally symmetric nonlinear beam equation, achieving $\dim = 9$ in the scalar and UW formulations and $\dim = 11$ in the full UVW-system.

The potential formulations reveal genuine potential symmetries — symmetries of the augmented systems that do not project to point symmetries of the original scalar equation. In particular, the logarithmic and fractional power constitutive laws appear as distinguished nonlinearities only in the potential systems and are entirely invisible to the standard Lie analysis of the fourth-order PDE.

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Chapter 1

Introduction

The study of elastic beams is a classical topic in applied mathematics and mechanics, originating in the eighteenth century through the work of Euler and Bernoulli. Their model describes the transverse motion of a slender beam and leads to the well-known Euler–Bernoulli equation

$$u_{tt} + a^2 u_{xxxx} = 0,$$

which remains a fundamental beam equation in structural mechanics. Despite its simplicity, this model captures essential features of bending and vibration, and it serves as a starting point for more general theories. In this thesis, we consider a nonlinear generalization of the Euler–Bernoulli model obtained from the Lagrangian

$$L(u_t, u_{xx}) = \frac{1}{2}u_t^2 - K(u_{xx}),$$

where K is a smooth function. The corresponding Euler–Lagrange equation is

$$u_{tt} + \left(K'(u_{xx})\right)_{xx} = 0,$$

which defines a family of nonlinear beam equations parameterized by the function K. The classical linear model is recovered when $K(z) = \frac{a^2}{2}z^2$. The purpose of this thesis is to analyze the structure of this nonlinear equation using symmetry and conservation-law methods. Rather than studying a single equation, we investigate the entire class and determine how its properties depend on the choice of the nonlinearity K. In particular, we aim to classify its Lie point symmetries and put the grounds to then identify its conservation laws, and use these structures together to obtain exact solutions and characterize distinguished nonlinear models.

In chapter 2, we derive the nonlinear beam equation from a variational principle and discuss its relation to the classical Euler–Bernoulli model, then chapter 3 introduces the symmetry framework we used. Chapter 4 introduces Jet spaces and their importance in understanding potential formalism and how it uncovers more symmetries while keeping the same physics of the PDE, and chapter 5 classifies the Lie point symmetries of the equation.

Chapter 2

The Nonlinear Beam Equation

This chapter introduces the nonlinear beam equation. We begin with the classical Euler–Bernoulli model, and replace its quadratic bending energy with a general constitutive function K, and derive the PDE we are interested in from a variational principle.

2.1 The classical Euler–Bernoulli model

Let $u(x,t)$ denote the transverse displacement of a thin elastic beam at position x and time t. Under the Euler–Bernoulli hypotheses [3, 8]—plane cross-sections remain plane and perpendicular to the centerline—the linearised equation of motion is

$$u_{tt} + a^2\, u_{xxxx} = 0, \qquad a > 0, \tag{2.1}$$

where a is a positive constant determined by material stiffness and cross-sectional geometry. The fourth-order spatial derivative arises because bending is governed by the curvature, which for small deflections is approximated by u_{xx}.

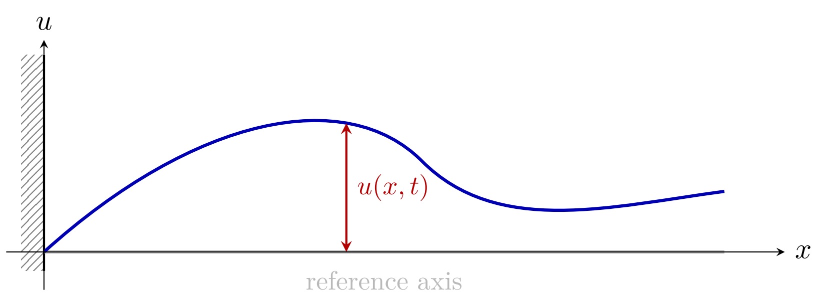

Figure 2.1: Transverse displacement $u(x,t)$ of a cantilever beam. The blue curve shows a representative deformed configuration.

2.2 A nonlinear bending energy

Equation (2.1) corresponds to a quadratic dependence of the stored elastic energy on curvature. To account for a more general material response, we replace the quadratic potential with an arbitrary smooth function K. Let $K \in C^2(\mathbb{R})$ satisfy $K'(z) \neq 0$ for all z in the domain of interest. The Lagrangian density [10] is

$$\mathcal{L}(u_t, u_{xx}) \;=\; \tfrac{1}{2}\,u_t^2 \;-\; K(u_{xx}), \tag{2.2}$$

where the first term is kinetic energy density and the second is stored bending energy density. This choice introduces a nonlinear constitutive law through the function K: the bending moment is proportional to $K'(u_{xx})$ rather than to u_{xx} itself.

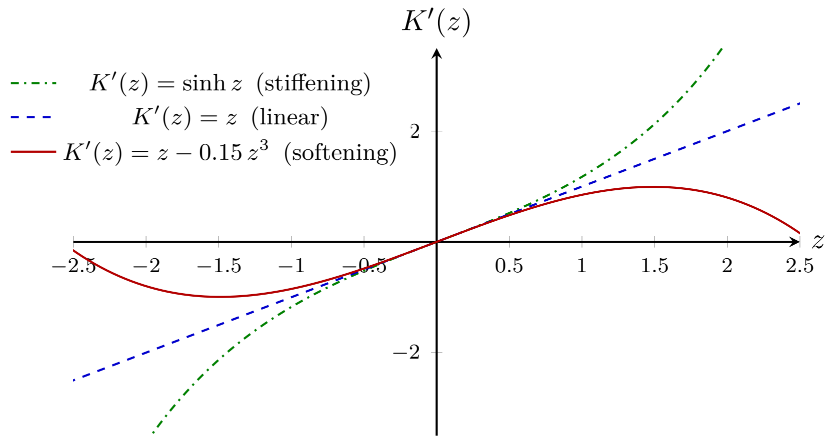

Figure 2.2: Three representative constitutive laws $K'(z)$. The dashed line represents the classical linear model. A softening nonlinearity (K' sub-linear for large $|z|$) and a stiffening nonlinearity (K' super-linear) are shown for comparison.

2.3 Derivation of the Euler–Lagrange equation

Consider the action functional

$$\mathcal{S}[u] \;=\; \iint \mathcal{L}(u_t, u_{xx})\,dx\,dt \;=\; \iint \Big(\tfrac{1}{2}\,u_t^2 - K(u_{xx})\Big)\,dx\,dt. \tag{2.3}$$

To derive the non-linear equation, we set $u \mapsto u + \varepsilon\,v$ where $v(x,t)$ is an arbitrary smooth function that vanishes on the boundaries of our integral. Differentiating at $\varepsilon = 0$ gives the first variation

$$\delta\mathcal{S} \;=\; \iint \big(u_t\,v_t - K'(u_{xx})\,v_{xx}\big)\,dx\,dt. \tag{2.4}$$

Integrating the first term by parts in t and the second term by parts twice in x (all boundary contributions vanish because v vanishes at the boundary) gives us

$$\delta\mathcal{S} \;=\; -\iint \Big[u_{tt} + \big(K'(u_{xx})\big)_{xx}\Big]\,v\;dx\,dt.$$

Since v is arbitrary, the fundamental lemma of the calculus of variations [10] gives the Euler–Lagrange equation

$$u_{tt} + \big(K'(u_{xx})\big)_{xx} = 0. \tag{2.5}$$

2.4 Recovery of the classical model

The classical beam equation is recovered by choosing a quadratic bending energy,

$$K(z) = \frac{a^2}{2} z^2. \tag{2.6}$$

Then $K'(z) = a^2 z$, and (2.5) reduces to

$$u_{tt} + a^2 u_{xxxx} = 0,$$

which is exactly (2.1). Hence the linear Euler–Bernoulli equation is a special case of the general nonlinear model.

2.5 Structure

We now examine the physical nature of (2.5):

$$\underbrace{u_{tt}}_{\text{inertia}} \; + \; \underbrace{\big(K'(u_{xx})\big)_{xx}}_{\text{nonlinear bending}} \; = \; 0. \tag{2.7}$$

So, what we are interested in is non-linear forms of K, and rather than studying a single equation, we study the whole family (2.5) parameterised by K, and our goal is to determine which choices of K lead to special symmetry or conservation properties.

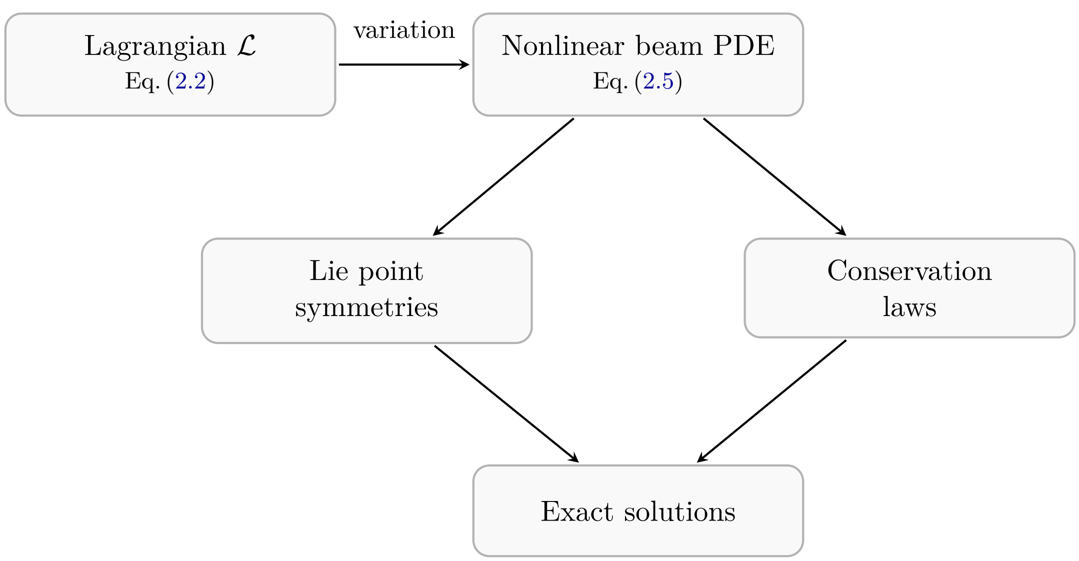

Figure 2.3: From the Lagrangian, to symmetries and conservation laws, to exact solutions that solve real problems.

Chapter 3

Symmetry Methods for Differential Equations

This chapter introduces the basic symmetry concepts used in the thesis needed for the nonlinear beam equation in one spatial dimension:

$$u_{tt} + \left(K'(u_{xx})\right)_{xx} = 0.$$

3.1 What is a symmetry?

A symmetry of a differential equation is a transformation that maps solutions to solutions of the same equation. For example, if $u(x,t)$ is a solution and replacing t by $t+\varepsilon$ produces another solution, then the equation has time-translation symmetry parametrized by ϵ. If shifting x by a constant preserves the equation, then it has space-translation symmetry. More generally, one considers a one-parameter family of point transformations [4]

$$\tilde{x} = x + \varepsilon\,\xi(x,t,u) + O(\varepsilon^2), \qquad \tilde{t} = t + \varepsilon\,\tau(x,t,u) + O(\varepsilon^2), \tag{3.1}$$

$$\tilde{u} = u + \varepsilon\,\phi(x,t,u) + O(\varepsilon^2), \tag{3.2}$$

if it preserves the given differential equation, we call it a one-parameter Lie group of symmetries parametrized by some small parameter ε, and $\tilde{u}|_{\epsilon=0} = u$, also called a Lie point symmetry.

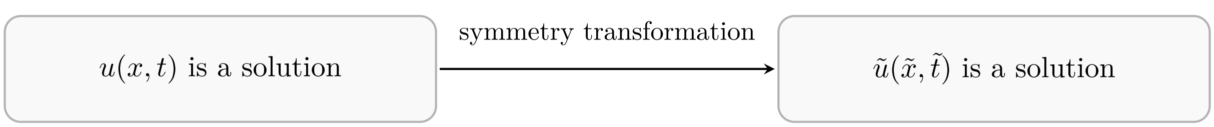

Figure 3.1: A symmetry maps solutions of the PDE to other solutions.

3.2 Infinitesimal generators

Instead of working with the full transformation, Lie theory studies its infinitesimal generator [4, 7],

$$X = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u. \tag{3.3}$$

The functions ξ, τ, and ϕ are unknown and must be determined from the invariance of the equation, and the advantage of using this framework is that the symmetry problem becomes algebraic and differential: one derives a system of determining equations for ξ, τ, and ϕ, then solves that system. Some standard examples are:

$$\partial_x \quad \text{(space translation)}, \qquad \partial_t \quad \text{(time translation)}, \qquad \partial_u \quad \text{(vertical translation)}.$$

Another reason why we adapt this framework is that using these infinitesimal generators give us a coordinate-independent way to characterize Lie symmetries, where X represents a tangent vector field in all coordinate systems. And if we regard $\{\partial_x, \partial_t, \partial_u\}$ as our basis for the space of vector fields, then X is the tangent vector field at (x, t, u). If such vector fields satisfy the invariance condition, then they generate symmetries of the PDE.

3.3 Why prolongation is needed

A differential equation depends not only on x, t, and u, but also on derivatives such as u_t, u_{xx}, and u_{tt}. Therefore, to test whether a transformation is a symmetry, one must know how it acts on derivatives as well. This extended action is called the *prolongation* of the vector field. For the beam equation, derivatives up to fourth order in x appear after expansion of

$$\big(K'(u_{xx})\big)_{xx},$$

so the symmetry generator must be prolonged to sufficiently high order. Schematically,

$$X \longrightarrow \operatorname{pr}^{(n)} X,$$

where $\operatorname{pr}^{(n)} X$ acts on all derivatives of u up to order n. For example, the first and second prolongations are:

$$\operatorname{pr}^{(1)} X = X + \phi^x \partial_{u_x} + \phi^t \partial_{u_t}, \tag{3.4}$$

$$\operatorname{pr}^{(2)} X = X + \phi^x \partial_{u_x} + \phi^t \partial_{u_t} + \phi^{xx} \partial_{u_{xx}} + \phi^{xt} \partial_{u_{xt}} + \phi^{tt} \partial_{u_{tt}}, \tag{3.5}$$

to compute the coefficients $\phi^x, \phi^t, \phi^{xx}, \ldots$, we first introduce the total derivative operators:

$$D_x = \partial_x + u_x \partial_u + u_{xx} \partial_{u_x} + u_{xt} \partial_{u_t} + ...$$

$$D_t = \partial_t + u_t \partial_u + u_{tt} \partial_{u_t} + u_{xt} \partial_{u_x} + ...$$

and from ξ, τ, ϕ:

$$\begin{aligned}
\phi^x &= D_x(\phi) - u_x D_x(\xi) - u_t D_x(\tau), \\
\phi^t &= D_t(\phi) - u_x D_t(\xi) - u_t D_t(\tau), \\
\phi^{xx} &= D_x(\phi^x) - u_{xx} D_x(\xi) - u_{xt} D_x(\tau), \\
\phi^{xt} &= D_t(\phi^x) - u_{xx} D_t(\xi) - u_{xt} D_t(\tau)
\end{aligned}$$

More generally [7], we can write:

$$\phi^J = D_{x_{j_1}}(\phi^{J'}) - \sum_k u_{J'k}\, D_{x_{j_1}}(\xi^k), \tag{3.6}$$

where J' is the multi-index J without its last entry, $u_{J'k}$ denotes the partial derivative obtained by appending the coordinate index k to J', and the sum runs over all independent variables (here $k = x, t$ so $\xi^x = \xi$, $\xi^t = \tau$). The recursion starts with $\phi^\emptyset = \phi$.

3.4 The invariance condition

Let

$$F(x, t, u^{(n)}) = 0 \tag{3.7}$$

denote a differential equation, where $u^{(n)}$ stands for all derivatives of u up to order n. Then a vector field X generates a Lie point symmetry if and only if

$$\mathrm{pr}^{(n)}\, X(F)\Big|_{F=0} = 0. \tag{3.8}$$

This is the infinitesimal invariance condition. For the nonlinear beam equation,

$$F := u_{tt} + \big(K'(u_{xx})\big)_{xx}. \tag{3.9}$$

The symmetry condition becomes

$$\mathrm{pr}^{(4)}\, X(F)\Big|_{F=0} = 0, \tag{3.10}$$

since fourth-order spatial derivatives are present. Equation (3.10) yields a system of linear partial differential equations for the unknown coefficients ξ, τ, and ϕ. These are called the determining equations, and that is exactly what we need to solve in order to solve the original PDE.

3.5 Determining equations and classification

The determining equations are solved by separating terms according to independent derivatives of u. This leads to restrictions on ξ, τ, and ϕ, and often also on the function K. There are two possible outcomes:

1. **Generic symmetries:** symmetries valid for arbitrary K;
2. **Extended symmetries:** additional symmetries that arise only for special choices of K.

This distinction is central to our classification problems, and our goal is to find all Lie point symmetries for all non-linear K families we can

$$u_{tt} + \big(K'(u_{xx})\big)_{xx} = 0$$

and to determine which nonlinearities K hide interesting symmetries.

3.6 Why symmetries matter

Symmetries are useful for several reasons. First, they reveal the geometric structure of the equation. Second, they can reduce a PDE to an ODE by introducing invariant variables. Third, they can produce exact solutions. Finally, in variational problems, symmetries are closely connected to conservation laws through Noether's theorem [6, 7].

Figure 3.2: Lie symmetries are one way to reduce a PDE to a set of ODEs.

Chapter 4

Jet Spaces

The objects defined in Chapter 3—prolonged vector fields, determining equations, invariance conditions—all live naturally in jet spaces. Making this setting explicit clarifies why introducing potential variables enlarges the symmetry algebra and how potential symmetries arise, and that's what we will talk about here.

4.1 Jet bundles

Let $x = (x^1, \ldots, x^p)$ denote the independent variables and $u = (u^1, \ldots, u^q)$ the dependent variables of a differential equation. The *n-th order jet space* [7, 9] $J^{(n)}(\mathbb{R}^p, \mathbb{R}^q)$ is the space whose coordinates consist of the independent variables, the dependent variables, and all partial derivatives of the dependent variables up to order n:

$$J^{(n)} = \{\, (x^i,\ u^\alpha,\ u^\alpha_J) \bigm| 1 \le i \le p,\ 1 \le \alpha \le q,\ 0 < |J| \le n \,\}, \tag{4.1}$$

where $J = (j_1, \ldots, j_k)$ is a multi-index of order $|J| = k$ and $u^\alpha_J = \partial^k u^\alpha / \partial x^{j_1} \cdots \partial x^{j_k}$. For a single function $u(x,t)$ of two variables ($p = 2$, $q = 1$), the zeroth jet space is simply $J^{(0)} = \{(x, t, u)\}$, the first jet space is

$$J^{(1)} = \{(x, t, u, u_x, u_t)\},$$

and each successive order adds one layer of derivatives. The dimension of $J^{(n)}(\mathbb{R}^2, \mathbb{R})$ is

$$\dim J^{(n)}(\mathbb{R}^2, \mathbb{R}) = 2 + \binom{n+2}{2} = 2 + \frac{(n+1)(n+2)}{2}. \tag{4.2}$$

For the fourth-order beam equation, the relevant space is $J^{(4)}(\mathbb{R}^2, \mathbb{R})$, which has dimension $2 + \binom{6}{2} = 17$.

4.2 Prolongation in jet space language

The prolongation operation introduced in Chapter 3 has a clean geometric meaning in jet space. A vector field on $J^{(0)}$,

$$X = \xi\, \partial_x + \tau\, \partial_t + \phi\, \partial_u,$$

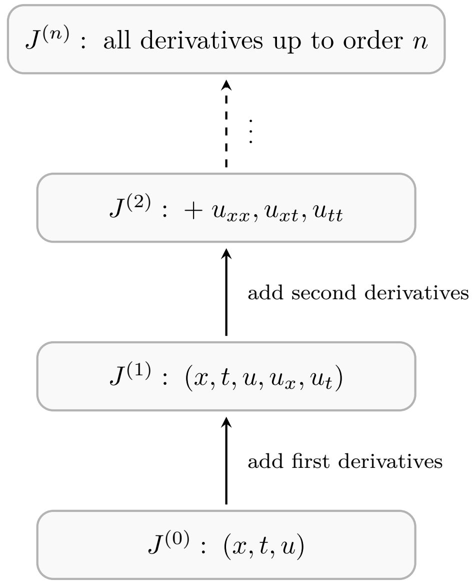

Figure 4.1: The tower of jet spaces for a scalar function $u(x,t)$. Each level adds a new layer of derivative coordinates.

acts on the base coordinates (x, t, u). To test whether it preserves a PDE that involves derivatives, we must lift X to a vector field on $J^{(n)}$ that is compatible with the contact structure (the requirement that the derivative coordinates genuinely represent derivatives of u). This lifted field is the n-th prolongation $\mathrm{pr}^{(n)} X$. The prolongation formulae from Chapter 3,

$$\phi^J = D_{j_1}(\phi^{J'}) - \sum_k u_{J'k}\, D_{j_1}(\xi^k),$$

are exactly the conditions ensuring that $\mathrm{pr}^{(n)} X$ respects the contact structure of $J^{(n)}$. The invariance condition $\mathrm{pr}^{(n)} X(F)\big|_{F=0} = 0$ is then a condition on a vector field in $J^{(n)}$, and the determining equations are their coordinate expression.

4.3 Changing the fibre dimension

The mechanism behind potential symmetries is a change of the fibre dimension q—the number of dependent variables. The four formulations studied here live in different jet spaces (we will see these in detail, but here V and W add no new information; they are merely used to write the equation in a new form):

System	**Dep. vars. (q)**	**Order (n)**	**Jet space**	$\dim J^{(n)}$
U	1	4	$J^{(4)}(\mathbb{R}^2, \mathbb{R})$	17
UV	2	3	$J^{(3)}(\mathbb{R}^2, \mathbb{R}^2)$	22
UW	2	3	$J^{(3)}(\mathbb{R}^2, \mathbb{R}^2)$	22
UVW	3	3	$J^{(3)}(\mathbb{R}^2, \mathbb{R}^3)$	32

The general dimension formula for q dependent variables in p independent variables at order n is

$$\dim J^{(n)}(\mathbb{R}^p, \mathbb{R}^q) = p + q \binom{n+p}{p}. \tag{4.3}$$

For our problem ($p = 2$):

$$\dim J^{(n)}(\mathbb{R}^2, \mathbb{R}^q) = 2 + q \, \frac{(n+1)(n+2)}{2}.$$

The key thing to note here is that even though the UV-system has lower differential order ($n = 3$ instead of $n = 4$), its jet space is larger than that of the U-system (22 vs. 17 coordinates) because the fibre has grown from $q = 1$ to $q = 2$. The UVW-system at $q = 3$ is larger still.

4.4 Why a larger jet space admits more symmetries

A symmetry generator on a system with q dependent variables has the form

$$X = \xi \, \partial_x + \tau \, \partial_t + \sum_{\alpha=1}^{q} \phi^\alpha \, \partial_{u^\alpha}.$$

The component functions ξ, τ, and $\phi^1, \ldots, \phi^q$ are the unknowns in the determining equations. Increasing q does two things simultaneously:

1. It adds new unknowns (ϕ^V, ϕ^W, etc.), giving the determining system more free parameters to satisfy.

2. It generates more determining equations (56 for the U-system vs. 140 for the UVW-system), imposing more constraints.

The net effect depends on the structure of the PDE system. For the nonlinear beam equation, the additional degrees of freedom consistently outweigh the additional constraints: the generic symmetry dimension increases from 7 (U) to 8 (UV) to 9 (UVW).

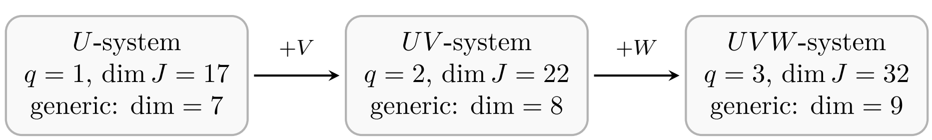

Figure 4.2: Increasing the fibre dimension by introducing potential variables enlarges the jet space and increases the generic symmetry algebra dimension.

4.5 Potential symmetries

A potential symmetry [1] of a PDE is a point symmetry of an augmented system (obtained by introducing potential variables) that does not project to a point symmetry of the original equation. In jet space language, this means: X is a well-defined vector field on $J^{(n)}(\mathbb{R}^2, \mathbb{R}^q)$ for $q > 1$ whose prolonged action preserves the augmented system, but no vector field on

$J^{(n')}(\mathbb{R}^2, \mathbb{R})$ produces the same action on solutions of the original scalar PDE. The mechanism is straightforward. A generator like

$$X_9 = \tfrac{1}{2}x^2\,\partial_U - \frac{Bx^3 + 3W}{6B}\,\partial_W + \frac{t}{B}\,\partial_t - \frac{x}{2B}\,\partial_x$$

(from the UW-system) depends explicitly on the potential variable W through its ∂_W component. Since W does not exist in the U-system, this generator has no counterpart there. It lives in $J^{(3)}(\mathbb{R}^2, \mathbb{R}^2)$ and cannot be projected down to $J^{(4)}(\mathbb{R}^2, \mathbb{R})$.

The classification in Chapter 5 identifies several such potential symmetries. The logarithmic and fractional-power constitutive laws, which appear as distinguished nonlinearities only in the UV-, UW-, and UVW-systems, are entirely invisible to the point-symmetry analysis of the scalar beam equation. The jet space framework explains why: the relevant vector fields simply do not exist in the smaller space.

4.6 Summary

The takeaways from this chapter are:

1. A PDE of order n in q dependent variables is a submanifold of $J^{(n)}(\mathbb{R}^p, \mathbb{R}^q)$.
2. Lie point symmetries are vector fields on $J^{(0)}$ whose n-th prolongation is tangent to this submanifold.
3. Introducing potential variables increases q, changes the ambient jet space, and can expose symmetries and conservation laws that do not exist in the smaller space.

Chapter 5

Lie Point Symmetries

We now do the complete Lie point symmetry classification of the nonlinear beam equation and three equivalent formulations obtained by introducing potential variables. For each formulation, the determining equations were generated and split using the `GeM` package in Maple [2, 5]; the over-determined systems were resolved with `rifsimp` / `casesplit`. We report the generic symmetries (valid for arbitrary K) and every special choice of K that enlarges the algebra. Degenerate branches yielding infinite-dimensional algebras (corresponding to linear or trivial K) are excluded throughout.

5.1 The four formulations

All four systems are equivalent representations of the same physics. We introduce them to change the jet space and expose potential symmetries: symmetries of an augmented system that do not show up as point symmetries of the original equation, and therefore cannot be discovered by the standard Lie analysis of the original PDE alone.

1. U**-system** (scalar, fourth order). The original nonlinear beam equation,

$$u_{tt} + \big(K'(u_{xx})\big)_{xx} = 0. \tag{5.1}$$

2. UV**-system** (two components, third order). We define a velocity potential $V(x,t)$ by $V_x = u_t$, lowering the temporal order at the cost of adding a dependent variable. The system then becomes

$$V_x - U_t = 0, \qquad V_t + \big(K'(U_{xx})\big)_x = 0. \tag{5.2}$$

3. UW**-system** (two components, third order). Define a momentum potential $W(x,t)$ through the conservation law relation $W_x = t\,U_t - U$. The system is

$$-t\,U_t + U + W_x = 0, \qquad W_t + t(K'(U_{xx}))_x = 0. \tag{5.3}$$

The potential W is conjugate to the scaling structure of the beam equation and captures a different conservation law from V.

4. UVW**-system** (three components, third order). Both potentials are introduced simultaneously. The system consists of four equations: the two compatibility conditions $V_x = U_t$, $W_x = t\,U_t - U$, together with evolution equations for V_t and W_t:

$$V_x - U_t = 0, \quad -t\,U_t + U + W_x = 0, \quad V_t + \left(K'(U_{xx})\right)_x = 0, \quad W_t + t(K'(U_{xx}))_x = 0. \tag{5.4}$$

Each formulation lives in a different (bigger) jet space than the original one. The symmetry generator then takes the general form

$$X = \xi\,\partial_x + \tau\,\partial_t + \phi^U\partial_U + (\phi^V\partial_V) + (\phi^W\partial_W),$$

where the terms in parentheses are present only when the corresponding variable is included in the formulation. A bigger jet space generally exhibits more symmetries.

5.2 The U-system

The scalar equation (5.1) produces 56 determining equations after prolongation to fourth order. We do a case-split, giving six branches: four with finite-dimensional algebras with non-linear $K'(z)$ and two degenerate (linear cases that we are not interested in).

5.2.1 Generic symmetries (arbitrary K)

For arbitrary K with $K'(z) \neq 0$, the symmetry algebra is seven-dimensional:

$$\begin{aligned} &X_1 = \partial_U, && X_2 = \partial_x, \quad X_3 = \partial_t, \\ &X_4 = x\,\partial_U, && X_5 = t\,\partial_U, \quad X_6 = xt\,\partial_U, \\ &X_7 = U\,\partial_U + t\,\partial_t + \tfrac{1}{2}x\,\partial_x. \end{aligned} \tag{5.5}$$

Generators X_1–X_6 are additive symmetries: they express the fact that adding any function $f(x,t)$ with $f_{xx} = 0$ to a solution produces another solution, since such functions contribute nothing to the bending term. The scaling symmetry X_7 reflects the dimensional balance $[x] : [t] = 1:2$ inherent in the equation, meaning that scaling space by a factor of 2, and time by a factor of 4 gives us the same solution space.

5.2.2 Special nonlinearities

Four special families of K extend the algebra.

Case 2 a: power law, $K(z) = (Az + B)^C$**.** An eighth generator appears:

$$X_8 = \tfrac{1}{2}x^2\,\partial_U - \frac{4A}{B}\,t\,\partial_t - \frac{A}{2B}\,x\,\partial_x. \tag{5.6}$$

Case 2 b: exponential, $K(z) = A\,e^{Bz}$**.** The extra generator simplifies to

$$X_8 = \tfrac{1}{2}x^2\,\partial_U - \tfrac{1}{2}B\,t\,\partial_t. \tag{5.7}$$

Case 3: reciprocal cube, $K(z) = -\frac{256A}{3(B+z)^3} + C$. Two extra generators bring the algebra to nine dimensions:

$$\begin{aligned} X_8 &= \tfrac{1}{2}(Bx^2 + 2U)\, t\, \partial_U + t^2\, \partial_t, \\ X_9 &= \tfrac{1}{2}x^2\, \partial_U + \frac{t}{B}\, \partial_t - \frac{x}{2B}\, \partial_x. \end{aligned} \tag{5.8}$$

Case 4: reciprocal, $K(z) = -\frac{4A}{B+z} + C$. One extra generator:

$$X_8 = \tfrac{1}{2}x^2\, \partial_U - \frac{x}{2B}\, \partial_x. \tag{5.9}$$

5.3 The UV-system

Introducing the velocity potential V via $V_x = U_t$ enlarges the dependent-variable space. In this case, we get 55 determining equations, and they split into five branches: three finite and two degenerate.

5.3.1 Generic symmetries (arbitrary K)

The generic algebra is eight-dimensional:

$$\begin{aligned} &X_1 = \partial_V, && X_2 = \partial_U, && X_3 = \partial_x, && X_4 = \partial_t, \\ &X_5 = x\, \partial_U, && X_6 = t\, \partial_U + x\, \partial_V, && X_7 = xt\, \partial_U + \tfrac{1}{2}x^2\, \partial_V, && \\ &X_8 = 2U\, \partial_U + V\, \partial_V + 2t\, \partial_t + x\, \partial_x. &&&&\end{aligned} \tag{5.10}$$

Compared with the U-system, the algebra gains one dimension. The new generator ∂_V shows translation freedom in the potential, and the U-system generators $t\, \partial_U$ and $xt\, \partial_U$ now carry V-components that enforce compatibility with $V_x = U_t$. The scaling generator X_8 is a little different from its U-system counterpart: the weights are $(U, V, t, x) \sim (2, 1, 2, 1)$; noting how scaling V by 2 requires scaling U by 4 to keep the PDE (same for t and x).

5.3.2 Special nonlinearities

Case 2 a: power law, $K(z) = (Az + B)^C$. Two extra generators show up (dimension 9):

$$\begin{aligned} X_8 &= \frac{Bx^2}{6A}\, \partial_U + V\, \partial_V - \tfrac{7}{6}t\, \partial_t - \tfrac{1}{6}x\, \partial_x, \\ X_9 &= \frac{-Bx^2 + 12AU}{12A}\, \partial_U + \tfrac{19}{12}t\, \partial_t + \tfrac{7}{12}x\, \partial_x. \end{aligned} \tag{5.11}$$

Case 2 b: exponential, $K(z) = A\, e^{Bz} + C$. Again, two extra generators (dimension 9):

$$\begin{aligned} X_8 &= \frac{x^2}{B}\, \partial_U + V\, \partial_V - t\, \partial_t, \\ X_9 &= \frac{2BU - x^2}{2B}\, \partial_U + \tfrac{3}{2}t\, \partial_t + \tfrac{1}{2}x\, \partial_x. \end{aligned} \tag{5.12}$$

Case 3: logarithmic, $K(z) = A\ln(B + Cz) + D$**.** One extra generator (dimension 9):

$$X_9 = \tfrac{1}{2}x^2\,\partial_U - \frac{3t}{B}\,\partial_t - \frac{3x}{B}\,\partial_x. \tag{5.13}$$

The logarithmic constitutive law is a special nonlinearity that appears in the UV-system classification with no counterpart in the scalar U-system, showing that introducing the velocity introduced a new model.

5.4 The UW-system

The momentum potential W, defined by $W_x = t\,U_t - U$, arises from a different conservation law of the beam equation. The system produces 56 determining equations and a richer case-split with seven branches: five finite-dimensional and two degenerate.

5.4.1 Generic symmetries (arbitrary K)

The generic algebra is seven-dimensional—the same count as the U-system, but with different generators:

$$\begin{aligned} &X_1 = \partial_W, && X_2 = \partial_x, && X_3 = t\,\partial_U, \\ &X_4 = xt\,\partial_U, && X_5 = x\,\partial_W - \partial_U, && X_6 = -x\,\partial_U + \tfrac{1}{2}x^2\,\partial_W, \\ &X_7 = \tfrac{2}{3}U\,\partial_U + W\,\partial_W + \tfrac{2}{3}t\,\partial_t + \tfrac{1}{3}x\,\partial_x. \end{aligned} \tag{5.14}$$

A key difference is that time translation ∂_t is absent from the generic algebra. The potential W is defined through the relation $W_x = tU_t - U$, which explicitly involves t; and time-translation invariance is broken at the generic level. The scaling weights in X_7 are $(U, W, t, x) \sim (\frac{2}{3}, 1, \frac{2}{3}, \frac{1}{3})$.

5.4.2 Special nonlinearities

Case 2 a: power law, $K(z) = (Az + B)^C + D$**.** One extra generator (dimension 8):

$$X_8 = -\frac{3Bx^2 + 2AU}{6B}\,\partial_U + \tfrac{1}{6}x^3\,\partial_W + \frac{25A}{6B}\,t\,\partial_t + \frac{A}{3B}\,x\,\partial_x. \tag{5.15}$$

Case 2 b: exponential, $K(z) = Ae^{Bz+C} + D$**.** One extra generator (dimension 8):

$$X_8 = -\tfrac{1}{2}x^2\,\partial_U + \tfrac{1}{6}x^3\,\partial_W + \tfrac{1}{2}B\,t\,\partial_t. \tag{5.16}$$

Case 3: fractional power, $K(z) = \frac{4A}{3}\big(\frac{-36B}{C+z}\big)^{1/3} + D$**.** One extra generator (dimension 8):

$$X_8 = \Big(-\tfrac{1}{2}x^2 - \tfrac{1}{24}U\Big)\partial_U + \tfrac{1}{6}x^3\,\partial_W + \tfrac{1}{24}x\,\partial_x. \tag{5.17}$$

This law is a cube-root type nonlinearity that does not appear in the U- or UV-system classifications.

Case 4: reciprocal cube, $K(z) = -\frac{256A}{3(B+z)^3} + C$ (maximal). The algebra reaches nine dimensions. Notably, the generic scaling generator X_7 is replaced by a modified scaling with different weights:

$$\begin{aligned} X_7 &= \tfrac{1}{2}(Bx^2 + 2U)\, t\, \partial_U + t^2\, \partial_t, \\ X_8 &= U\, \partial_U + \tfrac{3}{2}W\, \partial_W + t\, \partial_t + \tfrac{1}{2}x\, \partial_x, \\ X_9 &= \tfrac{1}{2}x^2\, \partial_U - \frac{Bx^3 + 3W}{6B}\, \partial_W + \frac{t}{B}\, \partial_t - \frac{x}{2B}\, \partial_x. \end{aligned} \tag{5.18}$$

The scaling weights in X_8 shift from the generic $(U, W, t, x) \sim (\frac{2}{3}, 1, \frac{2}{3}, \frac{1}{3})$ to $(1, \frac{3}{2}, 1, \frac{1}{2})$, and X_9 contains an explicit W-component, so this symmetry can't be in the original U-system.

Case 5: logarithmic, $K(z) = -\frac{1}{B}\ln(A + z) + C$. The generic scaling X_7 is again replaced, yielding two generators beyond X_1–X_6 (dimension 8):

$$\begin{aligned} X_7 &= (2Ax^2 + 2U)\, \partial_U + (-\tfrac{2}{3}Ax^3 + W)\partial_W - x\, \partial_x, \\ X_8 &= (-3Ax^2 - 2U)\, \partial_U + Ax^3\, \partial_W + t\, \partial_t + 2x\, \partial_x. \end{aligned} \tag{5.19}$$

As in the UV-system, the logarithmic constitutive law appears to emerge from the potential formulation.

5.5 The UVW-system

The full three-component system incorporates both potentials simultaneously. With 140 determining equations, this is the most computationally intensive formulation. The case-split yields six branches: four finite and two degenerate.

5.5.1 Generic symmetries (arbitrary K)

The generic algebra is nine-dimensional—the largest generic count among all four formulations:

$$\begin{aligned} &X_1 = \partial_W, && X_2 = \partial_V, && X_3 = \partial_x, \\ &X_4 = V\, \partial_W + \partial_t, && X_5 = x\, \partial_W - \partial_U, && X_6 = t\, \partial_U + x\, \partial_V, \\ &X_7 = -x\, \partial_U + \tfrac{1}{2}x^2\, \partial_W, && X_8 = xt\, \partial_U + \tfrac{1}{2}x^2\, \partial_V, \\ &X_9 = \tfrac{2}{3}U\, \partial_U + \tfrac{1}{3}V\, \partial_V + W\, \partial_W + \tfrac{2}{3}t\, \partial_t + \tfrac{1}{3}x\, \partial_x. \end{aligned} \tag{5.20}$$

The generator $X_4 = V\partial_W + \partial_t$ is noteworthy: time translation appears only when *coupled* to a W-shift proportional to V. This twist reflects the compatibility between the two potential definitions. The nine generic generators of the UVW-system contain, upon projection, all seven generic symmetries of the U-system, confirming consistency of the formulations.

5.5.2 Special nonlinearities

Case 2 a: power law, $K(z) = (Az + B)^C + D$. One extra generator (dimension 10):

$$X_{10} = -\frac{3Bx^2 + 2AU}{6B}\, \partial_U - \frac{11A}{3B}\, V\, \partial_V + \tfrac{1}{6}x^3\, \partial_W + \frac{11A}{3B}\, t\, \partial_t + \frac{A}{3B}\, x\, \partial_x. \tag{5.21}$$

Case 2 b: exponential, $K(z) = Ae^{Bz+C} + D$**.** One extra generator (dimension 10):

$$X_{10} = -\tfrac{1}{2}x^2\,\partial_U - \tfrac{1}{2}BV\,\partial_V + \tfrac{1}{6}x^3\,\partial_W + \tfrac{1}{2}B\,t\,\partial_t. \tag{5.22}$$

Case 3: reciprocal cube, $K(z) = -\frac{256A}{3(B+z)^3} + C$ The algebra reaches eleven dimensions — the overall maximum across all four formulations. This is the single most symmetric nonlinear model in the entire classification, and we note that both contain U, V, W, signaling that they are all needed to represent the algebra:

$$\begin{aligned} X_{10} &= -\tfrac{1}{2}(Bx^2 + 2U)\,t\,\partial_U + (-\tfrac{1}{6}Bx^3 + W)\partial_V - t^2\,\partial_t, \\ X_{11} &= -\frac{3Bx^2 + 2U}{6B}\,\partial_U + \frac{4V}{3B}\,\partial_V + \tfrac{1}{6}x^3\,\partial_W - \frac{4t}{3B}\,\partial_t + \frac{x}{3B}\,\partial_x \end{aligned} \tag{5.23}$$

Case 4: fractional power, $K(z) = \frac{4A}{3}\left(\frac{-36B}{C+z}\right)^3 + D$**.** The algebra is ten-dimensional. This case mirrors the fractional-power branch seen in the UW-system.

5.6 Discussion

Several things emerge from the classification.

Potential formalism enlarges the algebra. Introducing the velocity potential V raises the generic dimension from 7 to 8; introducing both V and W raises it to 9. The momentum potential W alone does not increase the generic count, but it does produce additional distinguished nonlinearities (the fractional power and logarithmic forms) not visible in the original equation.

Time translation can be broken or twisted. The UW-system and the UVW-system do not admit ∂_t as a generic symmetry. In the UW-system, time translation is non-existent altogether at the generic level because the potential definition $W_x = tU_t - U$ explicitly involves t. In the UVW-system, it reappears in the modified form $V\partial_W + \partial_t$, coupling the time shift to the potential.

New distinguished models arise from potentials. The logarithmic form of $K(z) = A\ln(B + Cz) + D$ is a distinguished nonlinearity in the UV- and UW-systems but not in the original equation. Similarly, the fractional-power form appears only in the UW- and UVW-systems. These are potential symmetries [1] in the sense that they are invisible to the point-symmetry analysis of the original fourth-order PDE.

System	$K(z)$	Point symmetries
U	Arbitrary	$X_1 = \partial_U,\ \ X_2 = \partial_x,\ \ X_3 = \partial_t,\ \ X_4 = x\,\partial_U,\ \ X_5 = t\,\partial_U,\ \ X_6 = xt\,\partial_U,$ $X_7 = U\partial_U + t\partial_t + \frac{1}{2}x\partial_x$
	$(Az+B)^C$	$X_8 = \frac{1}{2}x^2\partial_U - \frac{4A}{B}\,t\partial_t - \frac{A}{2B}\,x\partial_x$
	Ae^{Bz}	$X_8 = \frac{1}{2}x^2\partial_U - \frac{1}{2}B\,t\partial_t$
	$-\frac{4A}{B+z} + C$	$X_8 = \frac{1}{2}x^2\partial_U - \frac{x}{2B}\,\partial_x$
	$-\frac{256A}{3(B+z)^3} + C$	$X_8 = \frac{1}{2}(Bx^2 + 2U)\,t\,\partial_U + t^2\partial_t$ $X_9 = \frac{1}{2}x^2\partial_U + \frac{t}{B}\,\partial_t - \frac{x}{2B}\,\partial_x$
UV	Arbitrary	$X_1 = \partial_V,\ \ X_2 = \partial_U,\ \ X_3 = \partial_x,\ \ X_4 = \partial_t,\ \ X_5 = x\,\partial_U,$ $X_6 = t\,\partial_U + x\,\partial_V,\ \ X_7 = xt\,\partial_U + \frac{1}{2}x^2\partial_V,$ $X_8 = 2U\,\partial_U + V\,\partial_V + 2t\,\partial_t + x\,\partial_x$
	$(Az+B)^C$	$X_8 = \frac{Bx^2}{6A}\,\partial_U + V\partial_V - \frac{7}{6}t\partial_t - \frac{1}{6}x\partial_x$ $X_9 = \frac{12AU - Bx^2}{12A}\,\partial_U + \frac{19}{12}t\partial_t + \frac{7}{12}x\partial_x$
	$Ae^{Bz} + C$	$X_8 = \frac{x^2}{B}\,\partial_U + V\partial_V - t\partial_t$ $X_9 = \frac{2BU - x^2}{2B}\,\partial_U + \frac{3}{2}t\partial_t + \frac{1}{2}x\partial_x$
	$A\ln(B + Cz) + D$	$X_8 = \frac{1}{2}x^2\partial_U - \frac{3t}{B}\,\partial_t - \frac{3x}{B}\,\partial_x$
UW	Arbitrary	$X_1 = \partial_W,\ \ X_2 = \partial_x,\ \ X_3 = t\,\partial_U,\ \ X_4 = xt\,\partial_U,$ $X_5 = x\,\partial_W - \partial_U,\ \ X_6 = -x\,\partial_U + \frac{1}{2}x^2\partial_W,$ $X_7 = \frac{2}{3}U\,\partial_U + W\partial_W + \frac{2}{3}t\partial_t + \frac{1}{3}x\partial_x$
	$(Az+B)^C + D$	$X_8 = -\frac{3Bx^2 + 2AU}{6B}\,\partial_U + \frac{1}{6}x^3\partial_W + \frac{25A}{6B}\,t\partial_t + \frac{A}{3B}\,x\partial_x$
	$Ae^{Bz+C} + D$	$X_8 = -\frac{1}{2}x^2\partial_U + \frac{1}{6}x^3\partial_W + \frac{1}{2}B\,t\partial_t$
	$\frac{4A}{3}\left(\frac{-36B}{C+z}\right)^{1/3} + D$	$X_8 = \left(-\frac{1}{2}x^2 - \frac{1}{24}U\right)\partial_U + \frac{1}{6}x^3\partial_W + \frac{1}{24}x\,\partial_x$
	$-\frac{1}{B}\,\ln(A + z) + C$	$X_7 = (2Ax^2 + 2U)\,\partial_U + \left(-\frac{2}{3}Ax^3 + W\right)\partial_W - x\,\partial_x$ $X_8 = (-3Ax^2 - 2U)\,\partial_U + Ax^3\partial_W + t\,\partial_t + 2x\,\partial_x$
	$-\frac{256A}{3(B+z)^3} + C$	$X_7 = \frac{1}{2}(Bx^2 + 2U)\,t\,\partial_U + t^2\partial_t$ $X_8 = U\,\partial_U + \frac{3}{2}W\,\partial_W + t\,\partial_t + \frac{1}{2}x\,\partial_x$ $X_9 = \frac{1}{2}x^2\partial_U - \frac{Bx^3 + 3W}{6B}\,\partial_W + \frac{t}{B}\,\partial_t - \frac{x}{2B}\,\partial_x$
UVW	Arbitrary	$X_1 = \partial_W,\ \ X_2 = \partial_V,\ \ X_3 = \partial_x,\ \ X_4 = V\,\partial_W + \partial_t,$ $X_5 = x\,\partial_W - \partial_U,\ \ X_6 = t\,\partial_U + x\,\partial_V,$ $X_7 = -x\,\partial_U + \frac{1}{2}x^2\partial_W,\ \ X_8 = xt\,\partial_U + \frac{1}{2}x^2\partial_V,$ $X_9 = \frac{2}{3}U\,\partial_U + \frac{1}{3}V\,\partial_V + W\partial_W + \frac{2}{3}t\partial_t + \frac{1}{3}x\partial_x$
	$(Az+B)^C + D$	$X_{10} = -\frac{3Bx^2 + 2AU}{6B}\,\partial_U - \frac{11A}{3B}\,V\partial_V + \frac{1}{6}x^3\partial_W$ $\quad + \frac{11A}{3B}\,t\partial_t + \frac{A}{3B}\,x\partial_x$
	$Ae^{Bz+C} + D$	$X_{10} = -\frac{1}{2}x^2\partial_U - \frac{1}{2}B\,V\partial_V + \frac{1}{6}x^3\partial_W + \frac{1}{2}B\,t\partial_t$
	$-\frac{256A}{3(B+z)^3} + C$	$X_{10} = -\frac{1}{2}(Bx^2 + 2U)\,t\,\partial_U + (-\frac{1}{6}Bx^3 + W)\partial_V - t^2\partial_t$ $X_{11} = -\frac{3Bx^2 + 2U}{6B}\,\partial_U + \frac{4V}{3B}\,\partial_V + \frac{1}{6}x^3\partial_W$ $\quad - \frac{4t}{3B}\,\partial_t + \frac{x}{3B}\,\partial_x$

Table 5.1: Complete Lie point symmetry classification of the nonlinear beam equation across all four formulations. For each special K, only the additional generators beyond the generic basis are listed.

Chapter 6

Conclusion

6.1 Summary of results

We presented a systematic symmetry analysis of the nonlinear beam equation

$$u_{tt} + \big(K'(u_{xx})\big)_{xx} = 0$$

and three potential formulations derived from it. The main findings are as follows.

1. Complete symmetry classification. The Lie point symmetries were classified for all four systems (U, UV, UW, UVW) and for all nonlinear constitutive functions K. In the scalar equation, the generic algebra is seven-dimensional, spanned by space, time, and vertical translations, three affine symmetries, and a scaling symmetry. Four families of K enlarge this algebra: the power law $(Az+B)^C$, the exponential Ae^{Bz}, the reciprocal $-4A/(B+z)+C$, and the reciprocal cube $-256A/\big(3(B+z)^3\big)+C$. The reciprocal-cube model is the unique maximally symmetric nonlinear beam equation, admitting a nine-dimensional algebra.

2. Potential symmetries. Introducing velocity and momentum potentials changes the jet space in which the symmetry analysis takes place. Although the potential formulations are physically equivalent to the original equation, their larger fibre dimension admits vector fields that have no counterpart in the scalar jet space. The UV-system raises the generic dimension to eight, and the UVW-system to nine. The reciprocal-cube model achieves eleven dimensions in the UVW-system — the overall maximum. Two constitutive laws — the logarithmic form $A\ln(B+Cz)+D$ and the fractional-power form $\frac{4A}{3}(-36B/(C+z))^{1/3}+D$ — appear as distinguished nonlinearities only in the potential systems, confirming that they are genuine potential symmetries invisible to the standard point-symmetry analysis.

3. Structural observations. The classification revealed several structural features. Time translation ∂_t is a generic symmetry of the U- and UV-systems but is absent from the UW-system (because the potential definition $W_x = tU_t - U$ breaks time-translation invariance) and appears in the UVW-system only in the modified form $V\partial_W + \partial_t$. In the UW reciprocal-cube case, the generic scaling generator is replaced by a modified scaling with different weights, and the additional generator X_9 depends explicitly on W, providing a concrete example of a potential symmetry that cannot be projected to the original equation.

6.2 Future directions

Several natural extensions of this work present themselves.

Conservation laws. Since the beam equation is variational, Noether's theorem [6] links each variational symmetry to a conservation law. A systematic classification of conservation laws via the multiplier method — at zeroth, first, and second order — would complement the symmetry results and may reveal conserved quantities that are not so obvious, with potential formulations expected to yield additional conservation laws.

Exact solutions. The symmetries classified here can be used to reduce the nonlinear beam PDE to ordinary differential equations via invariant-variable methods. The reciprocal-cube model, with its maximal symmetry algebra, is the most natural candidate: its nine- or eleven-dimensional algebra offers a rich set of reductions. Travelling-wave, self-similar, and other group-invariant solutions would be of physical interest.

Higher-order and contact symmetries. This thesis restricted attention to Lie point symmetries. Extending the analysis to contact symmetries [7, 1] could uncover additional structure, particularly for the distinguished models identified here.

Bibliography

[1] George W. Bluman, Alexei F. Cheviakov, and Stephen C. Anco. *Applications of Symmetry Methods to Partial Differential Equations*. Vol. 168. Applied Mathematical Sciences. New York: Springer, 2010. ISBN: 978-0-387-98612-8. DOI: 10.1007/978-0-387-68028-6.

[2] Alexei F. Cheviakov. "GeM software package for computation of symmetries and conservation laws of differential equations". In: *Computer Physics Communications* 176.1 (2007), pp. 48–61. DOI: 10.1016/j.cpc.2006.08.001.

[3] Seon M. Han, Haym Benaroya, and Timothy Wei. "Dynamics of transversely vibrating beams using four engineering theories". In: *Journal of Sound and Vibration* 225.5 (1999), pp. 935–988. DOI: 10.1006/jsvi.1999.2257.

[4] Peter E. Hydon. *Symmetry Methods for Differential Equations: A Beginner's Guide*. Cambridge Texts in Applied Mathematics 22. Cambridge: Cambridge University Press, 2000. ISBN: 978-0-521-49786-3.

[5] Maplesoft. *Maple*. Waterloo Maple Inc., Waterloo, Ontario, Canada. 2024. URL: https://www.maplesoft.com/products/Maple/.

[6] Emmy Noether. "Invariante Variationsprobleme". In: *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* 1918 (1918), pp. 235–257.

[7] Peter J. Olver. *Applications of Lie Groups to Differential Equations*. 2nd. Vol. 107. Graduate Texts in Mathematics. New York: Springer-Verlag, 1993. ISBN: 978-0-387-95000-6. DOI: 10.1007/978-1-4612-4350-2.

[8] Singiresu S. Rao. *Vibration of Continuous Systems*. Hoboken, NJ: John Wiley & Sons, 2007. ISBN: 978-0-471-77171-5.

[9] David J. Saunders. *The Geometry of Jet Bundles*. Vol. 142. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press, 1989. ISBN: 978-0-521-36948-0. DOI: 10.1017/CBO9781107325883.

[10] Stephen T. Thornton and Jerry B. Marion. *Classical Dynamics of Particles and Systems*. 5th. Belmont, CA: Brooks/Cole–Thomson Learning, 2004. ISBN: 978-0-534-40896-1.